# Forced Orientation of Graphs

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**Abstract**

The concept of *forced orientation of graphs* was first introduced by Chartrand et al. in 1994. If, for a given assignment of directions to a subset $S$ of the edges of a graph $G$, there exists an orientation of $E(G) \setminus S$, so that the resulting graph is strongly connected, then that given assignment is said to be extendible to a strong orientation of $G$. A *forcing set* for a strong orientation $D$ of $G$ is a subset of $E(G)$, to which the assignment of orientations from $D$, can uniquely be extended to $E$ and thus result $D$. The size of the smallest forcing set for a strong orientation $D$ of $G$ is denoted by $f_D(G)$. In this note, we show that the family of all forcing sets for any particular strong orientation $D$ of $G$ is a matroid, and therefore all *minimal* forcing sets for $D$ have the same cardinality, $f_D(G)$. We also characterize those graphs $G$ that have strong orientations $D$, for which $f_D(G)$ is equal to the trivial maximum of $|E(G)|$.

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## 1 Introduction and preliminaries

In this paper, we consider only connected graphs. The set of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively, or by $V$ and $E$ when there is no ambiguity. For $S \subset V$, we denote by $[S, V \setminus S]$ the set of edges in $G$ that have exactly

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one endpoint in $S$ (regardless of the direction of the edge, in case $G$ is a mixed graph). We follow the definitions and notations of [13] for the concepts not defined here.

An *orientation* of a graph $G$ is a digraph $D$, with the same vertex set, whose underlying graph is $G$. A *strong orientation* is an orientation that is strongly connected, i.e., for any two vertices $u$ and $v$ there is a directed path from $u$ to $v$ and a directed path from $v$ to $u$.

A *partial orientation* of an undirected graph $G$ is a subset of the edges of an orientation of $G$. For a partial orientation $F$ of $G$, we define $G_F$ as the mixed graph whose underlying undirected graph is $G$ and its set of directed edges is precisely $F$. A partial orientation $F$ of $G$ is called *extendible* if there is a strong orientation $D$ of $G$ that contains $F$. A partial orientation $F$ is called a *strong orientation forcing set* or simply a *forcing set* for a strong orientation $D$ of $G$, if $D$ is the only strong orientation of $G$ which contains $F$. A *minimal forcing set* is a forcing set containing no other forcing set as a proper subset.

Notions similar to forcing sets are studied under different names of "defining sets" for combinatorial structures such as block designs [12] and graph colorings [7, 8, 9], and "critical sets" for Latin squares [1, 6, 7]. In [4], Chartrand et al. introduced and studied this notion for orientations of graphs. Here we take on this last concept and investigate some of the remaining problems.

The smallest number of edges in any forcing set for a strong orientation $D$ of $G$ is called the forcing number of $D$, and is denoted by $f_D(G)$. We also define $f(G)$ (also known as the forcing number of $G$) and $F(G)$ as the smallest and the largest values of $f_D(G)$, over all strong orientations $D$ of $G$. In [4], Chartrand et al. prove the following simple closed-form formula for $f(G)$.

**Theorem A** [4]**.** *If $G$ is a 2-edge-connected graph with $n$ vertices and $m$ edges, then $f(G) = m - n + 1$.*

The structure of this paper is as follows. In Section 2, we present definitions and general results that will be used throughout the paper. Section 3, studies the structure of forcing sets of a given strong orientation of a graph. Our main result of this section states that the family of the complements of forcing sets of a strong orientation is a matroid, and therefore every minimal forcing set of a strong orientation is also a smallest forcing set for that orientation. The results of section 4 give a characterization for those graphs $G$ for which $F(G) = |E(G)|$. Finally, we conclude with open problems in Section 5.

## 2 General results

In this section we state some useful results about orientations of graphs and their extensions. A well-known theorem on graph orientations is *Robbins' theorem*, which states that every 2-edge-connected undirected graph has a strong orientation (see [2]). In this paper, we use the following generalization of Robbins' theorem, due to Boesch and Tindell [3]. Notice that in the following, by a path in a mixed graph, we mean a path in which the direction of every directed edge conforms with the direction of the path.

**Theorem B** [3]**.** *Let $G$ be a mixed graph. The following statements are equivalent:*

**(a)** *The undirected edges of $G$ can be oriented in such a way that the resulting digraph is strongly connected.*

**(b)** *The underlying undirected graph of $G$ is 2-edge-connected and for every two vertices $u$ and $v$, there is a path from $u$ to $v$ and a path from $v$ to $u$.*

**(c)** *The underlying undirected graph of $G$ is 2-edge-connected and there is no subset $S$ of the vertices of $G$ such that all of the edges in $[S, V(G) \setminus S]$ are directed from $S$ to $V(G) \setminus S$.*

Theorem B leads us to the following definition.

**Definition.** *Let $F$ be a partial orientation of $G$, and $G_F$ denote the corresponding mixed graph. We say that an edge $e \in E(G)$ is* forced *by $F$, if there is a cut $[S, V \setminus S]$ in $G_F$ such that $e \in [S, V \setminus S]$ and all of the edges in $[S, V \setminus S]$, except $e$, are in $F$, and they are all directed in the same direction.*

The following lemma provides an equivalent definition for an edge being forced by a partial orientation.

**Lemma 1** *Let $F$ be an extendible partial orientation of $G$ and $e = uv$ be an edge in $E(G) \setminus F$. Then $e$ is forced by $F$ if and only if either there is no path from $u$ to $v$ or from $v$ to $u$ in $G_F - e$.*

**Proof.** If $e$ is forced by $F$, then for some $S \subset V$, $u \in S$ and $v \in V \setminus S$ and all of edges in $[S, V \setminus S]$, except $e$, are oriented by $F$ in the same direction, say, without loss of generality, from $S$ to $V \setminus S$. Then apparently, there is no path in $G_F - e$ from $v$ to $u$ since every edge incident to $V \setminus S$ is directed towards it.

Conversely, suppose there is no path from $u$ to $v$ in $G_F - e$. Let $S$ be the set of all vertices of $G$ to which there is a path from $u$ in $G_F - e$. Apparently $v \in V \setminus S$. Consider any edge $xy$ with $x \in S$ and $y \in V \setminus S$. If $F$ does not assign a direction to $xy$ or assigns the direction from $x$ to $y$, then the path from $u$ to $x$ can be extended to a path from $u$ to $y$ by adding $xy$ to it. But then $y$ must belong to $S$ and this contradicts our choice of $S$. Thus every edge $xy$ with $x \in S$ and $y \in V \setminus S$ must be oriented from $y$ to $x$ by $F$. ■

A very nice property of the forcing sets is their simultaneous "forcing" of the direction of every undirected edge of the graph. This is in contrast to the way most of the corresponding notions in other combinatorial contexts behave. For example, defining sets of graph colorings [7, 8, 9], do not necessarily force the color of every uncolored vertex at the same time and may instead only work in certain orders. The following theorem establishes this fact and is used in numerous places throughout this paper.

**Theorem 1** *An extendible partial orientation $F$ of $G$ is a strong orientation forcing set if and only if every edge $e \in E(G) \setminus F$ is forced by $F$.*

**Proof.** The "if" part is trivial. For the "only if" part, assume to the contrary that some edge $uv$ in $E(G) \setminus F$ is not forced by $F$. By Lemma 1, there are paths in $G_F - uv$ both from $u$ to $v$ and from $v$ to $u$. Thus, if we orient $uv$ in either direction, by Theorem B the resulting partial orientation can be extended into a strong orientation of $G$. But then, there is more than one way to extend $F$ into a strong orientation. ■

It is worth mentioning that the above theorem gives a polynomial time algorithm for recognizing forcing sets. This is in contrast to the result of Colbourn et al. [5] on the NP-hardness of recognizing critical sets in Latin squares.

## 3 The forcing set matroid

In this section we study the properties of forcing sets for any particular strong orientation of a graph. We will prove that the family of the complements of forcing sets for any orientation $D$ forms a matroid. This leads to an efficient algorithm for finding a smallest forcing set for a given strong orientation. At the heart of our proof is the following definition of a binary relation "$\preceq$" between the edges of a digraph.

**Definition.** *For any two edges $e_1$ and $e_2$ of a strongly connected digraph $D$, $e_1 \preceq e_2$ if every directed cycle $C$ of $D$ containing $e_1$ also contains $e_2$. Moreover, we write $e_1 \approx e_2$ if $e_1 \preceq e_2$ and $e_2 \preceq e_1$.*

It is easy to see that the relation $\preceq$ is a preorder, i.e., it is reflexive and transitive. This implies that the relation $\approx$ is an equivalence relation and thus partitions the set of edges of $D$ into equivalence classes. The relation $\preceq$ induces a partial order among the equivalence classes of $\approx$. The following two lemmas give a characterization of these equivalence classes.

**Lemma 2** *In a strongly connected digraph $D$ we have $e_1 \preceq e_2$ if and only if there is a cut $[S, V \setminus S]$ such that $e_1$ is from $S$ to $V \setminus S$, $e_2$ from $V \setminus S$ to $S$, and every other edge in the cut is from $S$ to $V \setminus S$.*

**Proof.** The "if" part is trivial. For the "only if" part, let $e_1 = uv$ and suppose $e_1 \preceq e_2$. If there exists a path from $v$ to $u$ in $D - e_2$, this path together with $e_1$, would make a cycle containing $e_1$ but not $e_2$, contradicting the assumption that $e_1 \preceq e_2$. Now, let $S$ be the set of vertices that are *not* reachable from $v$ in $D - e_2$. Then, $u \in S$ and $v \in V \setminus S$, and every edge in $[S, V \setminus S]$ except $e_2$, is directed from $S$ to $V \setminus S$. On the other hand, $D$ is strongly connected and thus $e_2$ must be directed from $V \setminus S$ to $S$. ■

**Lemma 3** *Let $D$ be a strongly connected digraph. For any two edges $e_1$ and $e_2$ in $D$, $e_1 \approx e_2$ if and only if $\{e_1, e_2\}$ is a cut set.*

**Proof.** By Lemma 2 we know that there exits a cut $[S, V \setminus S]$ containing both $e_1$ and $e_2$ such that all of its edges except $e_2$ are directed from $S$ to $V \setminus S$. We claim that $[S, V \setminus S]$ does not contain any edges other than $e_1$ and $e_2$. Assume to the contrary that there exists an edge $uv$ in $[S, V \setminus S]$ other than $e_1$ and $e_2$. Strong connectivity of $D$ implies that there is a path $P_1$ from the head of $e_2$ to $u$. This path cannot pass through $V \setminus S$ since the only edge from $V \setminus S$ to $S$ is $e_2$. Similarly, there is a path $P_2$ in $V \setminus S$ from $v$ to the tail of $e_2$. The two paths $P_1$ and $P_2$ along with $e_2$ and $uv$ form a cycle which contains $e_2$, but not $e_1$ and this is a contradiction. ■

**Lemma 4** *Let $e_1$ and $e_2$ be two edges in a strongly connected digraph $D$ such that $e_2 \not\preceq e_1$. If $F$ is a forcing set for $D$ containing $e_2$ but not $e_1$, then $e_1$ is forced by $F \setminus \{e_2\}$. In other words, if we remove an edge $e$ from a forcing set $F$ of a strongly connected digraph $D$, then the set of edges that are not forced by $F \setminus \{e\}$ is a subset of the set $\{x \in E \mid e \preceq x\}$.*

**Proof.** By Theorem 1, there is a cut $[S, V \setminus S]$ containing $e_1$, such that every edge of this cut, except $e_1$ belongs to $F$ and is directed from $S$ to $V \setminus S$ while $e_1$ is directed from $V \setminus S$ to $S$. If $e_2 \notin [S, V \setminus S]$ we are done. Otherwise, if $e_2 \in [S, V \setminus S]$, then by Lemma 2 we obtain $e_2 \preceq e_1$, a contradiction. ■

**Lemma 5** *Let $D$ be a strongly connected digraph and $e_1$ and $e_2$ be two edges of $D$ such that $e_1 \preceq e_2$. If $F$ is a forcing set for $D$, then $F \cup \{e_1\} \setminus \{e_2\}$ is also a forcing set for $D$.*

**Proof.** It is sufficient to prove that $F \cup \{e_1\} \setminus \{e_2\}$ forces the direction of $e_2$. Assume to the contrary that this does not happen. By Lemma 2, we know that there is a cut $[S, V \setminus S]$, containing $e_1$ and $e_2$, so that all of its edges except $e_2$ are directed toward $S$. Let $e \in [S, V \setminus S]$ be an edge other than $e_2$. If $F \setminus \{e_2\}$ does not force the direction of $e$, then by Lemma 4, we have $e_2 \preceq e$. On the other hand, by Lemma 2, we have $e \preceq e_2$. This means that $e \approx e_2$. An argument like the one in the proof of Lemma 3 shows that

$[S, V \setminus S] = \{e, e_2\}$. But since $e_1 \in [S, V \setminus S]$, $e$ cannot be any edge other than $e_1$. Thus every edge in $[S, V \setminus S] \setminus \{e_2\}$ is forced by $F \cup \{e_1\} \setminus \{e_2\}$. This, together with the fact that $e_2$ is the only edge in $[S, V \setminus S]$ directed toward $V \setminus S$, show that the direction of $e_2$ is forced by the set $F \cup \{e_1\} \setminus \{e_2\}$. Thus, $F \cup \{e_1\} \setminus \{e_2\}$ is a forcing set. ■

An equivalence class $C$ of the relation $\preceq$ is called *minimal* if for no two edges $e \in C$ and $e' \in E \setminus C$, we have $e' \preceq e$.

**Lemma 6** *Let $D$ be a strongly connected digraph and $F$ be an arbitrary forcing set for $D$. Then, any minimal equivalence class under the relation $\preceq$, must have at least one edge in $F$.*

**Proof.** Assume to the contrary that $C$ is a minimal equivalence class of the relation $\preceq$ and none of its elements belong to the forcing set $F$. Let $e$ be an edge in $C$. Then $e$ must be forced by $F$. So, by Theorem 1, there exists a cut $[S, V \setminus S]$ which contains $e$ and all of its edges except $e$ are directed toward $S$ and are in $F$. Let $e'$ be an edge in $[S, V \setminus S] \setminus \{e\}$. By Lemma 2, $e' \preceq e$. Also, we know that $e' \notin C$, because $e' \in F$ and $F \cap C = \emptyset$. This contradicts the minimality of the class $C$ under the relation $\preceq$. ■

The following theorem characterizes the set of all minimal forcing sets of a given strong orientation.

**Theorem 2** *A subset $F$ of the edges of a strongly connected digraph $D$ is a minimal forcing set for $D$ if and only if $F$ contains exactly one edge from each equivalence class of the relation $\approx$ which is minimal under the relation $\preceq$.*

**Proof.** Follows directly from Lemma 5 and Lemma 6. ■

We are now ready to state the main result of this section.

**Theorem 3** *For every strongly connected digraph $D$, the family $\mathcal{M}$ of subsets of the edges of $D$ whose complement is a forcing set for $D$ is a matroid.*

**Proof.** It is clear that for every $A \in \mathcal{M}$, every subset of $A$ is also in $\mathcal{M}$. Therefore, to show that $\mathcal{M}$ is a matroid, we only need to verify the *exchange property* [10]: for any two sets $A, B \in \mathcal{M}$, if $|A| < |B|$, there is an element $e \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{M}$. By the definition of $\mathcal{M}$, this statement is equivalent to the following: for any two forcing sets $\bar{A}$ and $\bar{B}$ for $D$, if $|\bar{A}| > |\bar{B}|$, then there is an edge $e \in \bar{A} \setminus \bar{B}$ such that $\bar{A} \setminus \{e\}$ is a forcing set for $D$. We prove this statement as follows: since equivalence classes of $\approx$ partition the set of edges of $D$ and $|\bar{A}| > |\bar{B}|$, there must be at least one equivalence class $C$ of $\approx$ which contains more elements of $\bar{A}$ than $\bar{B}$. Therefore, there must be at least one element $e$ in $C$ that belongs to $\bar{A}$ but not $\bar{B}$. We claim that $\bar{A} \setminus \{e\}$ is still a forcing set of $D$. By Theorem 2, we only need to show that for every equivalence class $C'$ of $\approx$, if $C'$ is minimal under $\preceq$, then it contains at least one element of $\bar{A} \setminus \{e\}$. Since $\bar{A}$ is a forcing set and therefore satisfies this condition, we only need to verify the above condition for the class $C' = C$ from which the element $e$ is removed. However, we know that $C$ contains more elements of $\bar{A}$ than $\bar{B}$, and therefore even after removal of $e$, $C$ contains at least as many elements of $\bar{A} \setminus \{e\}$ as those of $\bar{B}$. Hence, by Theorem 2 and using the fact that $\bar{B}$ is a forcing set, $C$ contains at least one element of $\bar{A} \setminus \{e\}$. Hence $\bar{A} \setminus \{e\}$ is a forcing set. This completes the proof of the theorem. ■

The above theorem assigns a matroid to every strongly connected digraph. A natural question will then be whether this matroid is related to the other known matroids on graphs.

The following is an immediate corollary of the above theorem.

**Corollary 1** *Let $D$ be a strong orientation of a graph $G$. Then all minimal forcing sets for $D$ have the same size.*

Note that Corollary 1 yields an efficient algorithm for constructing the smallest forcing set for a given strong orientation of a graph: the algorithm starts with the forcing set that consists of all edges of the graph, and iteratively removes edges that are forced by the remaining edges in the set, until it finds a minimal forcing set. By Corollary 1, any minimal forcing set is a forcing set of smallest size.

## 4 Orientations with a large forcing number

The forcing number of any orientation $D$ of a graph $G$ with $n$ vertices and $m$ edges is lower bounded by $m - n + 1$ (by Theorem A) and upper bounded by $m$. In this section, we give a simple characterization of graphs for which there is an orientation that attains this upper bound. In other words, we characterize graphs $G$ with $F(G) = m$.

**Lemma 7** *Let $D$ be a strong orientation of an undirected graph $G$ with $m$ edges. Then $f_D(G) = m$ if and only if for every edge $e$, $D - e$ is strongly connected.*

**Proof.** To prove sufficiency, assume to the contrary that $D$ has a forcing set $F$ of size strictly less than $m$. Let $e$ be an edge not in $F$. Now, since $D - e$ is strongly connected, $e$ is not forced by $F$. But this is in contradiction with Theorem 1.

For the necessity, assume that there is an edge $e$ in $D$, such that $D - e$ is not strongly connected and thus there is no path in $D - e$ from one of the endpoints of $e$ to the other. But this implies that $e$ is forced by $D - e$ and therefore $D - e$ is a forcing set of size $m - 1$ for $D$. ■

According to the terminology of [13], a digraph $D$ is *$i$-strongly connected*, if for any set $S$ of $i-1$ edges of $D$, the graph $D \setminus S$ is strongly connected. Using this definition, Lemma 7 can be restated as "$f_D(G) = m$ if and only if $D$ is a 2-strongly connected orientation of $G$."

The next theorem gives a necessary and sufficient condition for a graph to have a 2-strong orientation.

**Theorem 4** *For a graph $G$ with $m$ edges, $F(G) = m$ if and only if $G$ is 4-edge-connected.*

**Proof.** Assume that $G$ is 4-edge-connected. By a theorem of Nash-Williams (see for example [13]) this implies that $G$ has a 2-strong orientation $D$. Therefore by Lemma 7, $f_D(G) = m$, implying $F(G) = m$.

Now suppose $F(G) = m$ and let $D$ be a strong orientation of $G$ for which $f_D(G) = m$ and suppose that $G$ has a cut set $[S, V \setminus S]$ of size 3 or smaller. Each of the three edges of this cut are either directed from $S$ to $V \setminus S$, or from $V \setminus S$ to $S$. Since $D$ is a strong orientation, not all of these three edges can agree in their directions and thus exactly one, which we call $e$, must disagree with the other two. However, this means that $D - e$ is not strongly connected and therefore $e$ is forced by $D - e$. Thus, by Lemma 7, $f_D(G) < m$, a contradiction. ■

## 5 Conclusion and open problems

The main result of this paper was a nice characterization of forcing sets of a particular orientation of a graph, leading to polynomial time algorithms for recognizing forcing sets and finding minimal forcing sets in a digraph.

A family of problems, analogous to those considered in this papers, can be introduced by replacing strong orientations with *unilateral orientations* in the definition of forcing sets (see [4, 11]). A *unilateral orientation* of a graph $G$, is an orientation of $G$ in which for every pair of vertices $u, v \in V(G)$, there exists either a path from $u$ to $v$, or one from $v$ to $u$ (or both). Many of the problems regarding unilateral forced orientations are open. For example, we do not know of any efficient algorithm for recognizing unilateral forcing sets, or finding the smallest unilateral forcing set in a given digraph.

Another open problem is to find a simple way to compute $F(G)$ for a given undirected graph $G$. Results of Section 4 solve this problem for 4-edge-connected graphs. For graphs of edge-connectivity 2 and 3 this problem is widely open.

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