Computing equilibria in a Fisher market with linear single-constraint production units

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Abstract. We study the problem of computing equilibrium prices in a Fisher market with linear utilities and linear single-constraint production units. This setting naturally appears in ad pricing where the sum of the lengths of the displayed ads is constrained not to exceed the available ad space. There are three approaches to solve market equilibrium problems: convex programming, auction-based algorithms, and primal-dual. Jain, Vazirani, and Ye recently proposed a solution using convex programming for the problem with an arbitrary number of production constraints. A recent paper by Kapoor, Mehta, and Vazirani proposes an auction-based solution. No primal-dual algorithm is proposed for this problem.

In this paper we propose a simple reduction from this problem to the classical Fisher setting with linear utilities and without any production units. Our reduction not only imports the primal-dual algorithm of Devanur et al. to the single-constraint production setting, but also: i) imports other simple algorithms, like the auction-based algorithm of Garg and Kapoor, thereby providing a simple insight behind the recent sophisticated algorithm of Kapoor, Mehta, and Vazirani, and ii) imports all the nice properties of the Fisher setting, for example, the existence of an equilibrium in rational numbers, and the uniqueness of the utilities of the agents at the equilibrium.

1 Introduction

The model. Consider a market with m buyers and n sellers, each seller offering one unit of a good for sale. Each buyer i has a budget B_i, and a utility function u_i that specifies the utility of the buyer for each bundle of goods. Throughout this paper, we assume that u_i's are linear, i.e., the utility of i can be written as $\sum_j u_{ij}x_{ij}$, where u_{ij} is a non-negative number indicating the utility of buyer i for good j (the good sold by the j'th seller), and x_{ij} is the amount of good j bought by i. A market equilibrium is a price vector $\mathbf{p} \in \mathbb{R}^n$ and an allocation $\mathbf{x}$ of goods to buyers such that:

(i) The allocation maximizes the utility of each buyer at the current prices subject to his/her budget. More precisely, for every buyer i, the vector $\mathbf{x}$ is a solution of the maximization program

$$\begin{aligned} \text{maximize} \quad & \sum_j u_{ij} x_{ij} \\ \text{subject to} \quad & \sum_j p_j x_{ij} \le B_i. \end{aligned}$$

(ii) The demand and the supply for each good are equal, i.e., for every good j, $\sum_i x_{ij} = 1$. Note that as a corollary, we get that the budget of every buyer is also completely spent in buying the allocated bundle of goods.

The setting defined above is a special case of a more general setting formulated by Arrow and Debreu [1]. One of the features of the model by Arrow and Debreu is that it allows for *production units*. In this paper, we consider a very simple form of production units, defined as follows: in the Fisher model defined above, assume instead of the sellers, we have l production units. The k'th production unit has the capability of producing any bundle of goods $\mathbf{y}_k \in \mathbb{R}^n$ that satisfies a single linear constraint of the form $\mathbf{a}_k.\mathbf{y}_k \le 1$. Clearly, this model is a generalization of the Fisher model defined above, since each seller in the Fisher setting can be viewed as a production unit producing a single good. Linear single-constraint production units appear in settings like advertisement pricing [5], where each good corresponds to a placement of an advertisement, and each producer corresponds to a page that can "produce" any combination of ad placements whose hieghts sum to any value less than or equal to the height of the ad space.

A market equilibrium in the Fisher model with production units is given by a price vector $\mathbf{p} \in \mathbb{R}^n$ (p_j is the price of the j'th good) and an allocation $\mathbf{x} \in \mathbb{R}^{m \times n \times l}$ (x_{ijk} is the amount of the j'th good purchased from the k'th seller by the i'th buyer) satisfying the following conditions:

- At these prices, each buyer is allocated with a bundle maximizing his/her utility subject to his/her budget.
- At these prices, each producer is asked to produce a bundle maximizing his/her revenue subject to his/her production constraint.
- Total quantity of each good produced is the same as the total quantity of each good bought by the buyers (this condition is covered by the previous two conditions because of our choice of notation, for clarity we still write it as a separate condition).

The model of linear single-constraint production units that we consider in this paper is very restrictive, and might be considered too simplistic to be practical. However, even in this simple model, there are relatively complicated combinatorial algorithms proposed to compute the market equilibrium prices [7]. The point of this paper is to show that such algorithms can be obtained through a simple and intuitive reduction to the setting without production units, and then using any of the numerous algorithms proposed for that setting [3, 2, 4].

2 The reduction

The idea of the reduction is simple. We can view each production unit k as a unit that originally owns one unit of "raw material", and can transform this raw material to any bundle of goods satisfying $\mathbf{a}^k.\mathbf{y}^k \leq 1$. In order to reduce the model with production units to the simple buyer-seller setting, we construct a market in which each seller is selling the raw materials, and buyers directly buy the raw materials and convert them to their desired goods. We then give a simple transformation that converts the prices for raw materials to prices for the goods.

An alternative way to see the intuition behind our reduction is through the Eisenberg-Gale convex program [3]. This convex program which was originally formulated to solve the Fisher problem without production and later extended [6] to solve the Fisher problem with production. These convex programs intuitively say that among all feasible productions and allocations of the goods to the buyers, an equilibrium choses the production and allocation which maximizes the budget-weighted geometric mean of the buyers' utilities. Note that the convex program does not specify who does the production. In the setting of this paper, sellers produce finished goods from the raw material. Our reduction is that buyers buy the raw material and produce those finished goods which are most beneficial to them. Clearly, the set of feasible productions and allocations remains the same. Hence, the convex program of [6] does not really change. Therefore, the solution to both instances remain the same. To convert it into the classical Fisher setting with linear utilities and without production, we further show that the buyers can dissolve their production constraint into their utility functions - hence no production remains.

We now give a precise formulation of our reduction. Given an instance M of the market equilibrium problem with production units with m buyers, n goods, and l producers described by budgets $\mathbf{B} \in \mathbb{R}^m$, utilities $\mathbf{u} \in \mathbb{R}^{m \times n}$, and production constrains $\mathbf{a} \in \mathbb{R}^{l \times n}$, we construct another instance M' as follows: in M' we have m buyers (each corresponding to a buyer in M), and l sellers (each corresponding to a production unit in M) each offering one unit of a different good. We call the good sold by the k'th seller *the k'th raw material*. The budget of buyer i is B_i, and her utility for one unit of the k'th raw material is $u'_{ik} := \max_j \frac{u_{ij}}{a_{kj}}$.

Let $\mathbf{p}'$ denote the equilibrium price vector for the market M', and $\mathbf{x}'$ be a corresponding allocation. Such an equilibrium is guaranteed to exist and can be computed using any of the algorithms proposed for the linear Fisher model [3, 2, 4]. We define a price vector $\mathbf{p}$ for the goods in M by letting $p_j := \min_k p'_k.a_{kj}$. We now show that this price vector induces an equilibrium in M. To do this, we construct an allocation $\mathbf{x}$.

For every buyer i and seller k in M', take a good j that maximizes u_{ij}/a_{kj}, and define $x_{ijk} := x'_{ik}/a_{kj}$. For every other j, define $x_{ijk} := 0$. The following two lemmas show that the price vector $\mathbf{p}$ with the allocation $\mathbf{x}$ form a market equilibrium for M.

Lemma 1. *For every buyer i^*, the bundle given by the allocation $\mathbf{x}$ to i^* is an optimal feasible bundle for i^* at prices $\mathbf{p}$.*

Proof. First, we verify that i^* can afford the bundle given to her by $\mathbf{x}$. For every production unit k, by the definition of $\mathbf{x}$, $\sum_j p_j x_{i^*jk} = p_{j^*} x'_{i^*k}/a_{kj^*}$, where j^* is a good maximizing u_{i^*j}/a_{kj} over all j. Therefore, by definition of $\mathbf{p}$, $\sum_j p_j x_{i^*jk} = p'_k . a_{kj^*} . (x'_{i^*k}/a_{kj^*}) = p'_k x'_{i^*k}$. Hence, $\sum_{k,j} p_j x_{i^*jk} = \sum_k p'_k x'_{i^*k} \leq B_{i^*}$.

To show the optimality of this bundle, we need to show that for every j^*, k^*, if $x_{i^*j^*k^*} > 0$, then the good j^* maximizes the ratio u_{i^*j}/p_j over all j. By the definition of $\mathbf{x}$, $x_{i^*j^*k^*} > 0$ implies that $j^* \in \text{argmax}_j\{u_{i^*j}/a_{k^*j}\}$ and $x'_{i^*k^*} > 0$. Since $(\mathbf{p}', \mathbf{x}')$ is a market equilibrium in M', the latter inequality implies that $k^* \in \text{argmax}_k\{u'_{i^*k}/p'_k\}$. By combining these two statements and the definition of $\mathbf{u}'$, we can conclude that $(j^*, k^*) \in \text{argmax}_{(j,k)}\{\frac{u_{i^*j}}{a_{kj}p'_k}\}$, or equivalently, that j^* maximizes $\max_k\{\frac{u_{i^*j}}{a_{kj}p'_k}\} = \frac{u_{i^*j}}{\min_k(a_{kj}p'_k)} = u_{i^*j}/p_j$ over all j.

Lemma 2. *For every production unit k^*, the bundle given by $\mathbf{x}$ is the optimal feasible bundle that k^* can produce.*

Proof. We start by showing that the bundle given by $\mathbf{x}$ can be produced under k^*'s production constraint. For every buyer i, by the definition of $\mathbf{x}$, $\sum_j a_{k^*,j} x_{ijk^*} = x'_{ik^*}$. Therefore, $\sum_{i,j} a_{k^*,j} x_{ijk^*} = \sum_i x'_{ik^*} \leq 1$, since there is exactly one unit of the k^*'th raw material in M'.

To show the optimality of the bundle, we need to show that for every i^*, j^*, if $x_{i^*j^*k} > 0$, then the good j^* maximizes the ratio p_j/a_{k^*j} over all j. By the argument in the proof of the previous lemma, $x_{i^*j^*k} > 0$ implies that $k^* \in \text{argmax}_k\{\frac{u_{i^*j^*}}{a_{kj^*}p'_k}\}$. Therefore, $k^* \in \text{argmin}_k\{a_{kj^*}p'_k\}$. By the definition of $\mathbf{p}$, this means that $p_{j^*} = a_{k^*j^*}p'_{k^*}$. Therefore, for every good j,

$$\frac{p_j}{a_{k^*j}} = \frac{\min_k\{p'_k a_{kj}\}}{a_{k^*j}} \leq \frac{p'_{k^*}a_{k^*j}}{a_{k^*j}} = \frac{p'_{k^*}a_{k^*j^*}}{a_{k^*j^*}} = \frac{p_{j^*}}{a_{k^*j^*}},$$

as desired.

The above lemmas show that $(\mathbf{p}, \mathbf{x})$ is an equilibrium for M. Therefore, we have the following theorem.

Theorem 1. *For every instance M of the market equilibrium problem with linear single-constraint production units, there is an instance M' of the market equilibrium problem without production units such that*

- *Given M, M' can be constructed in polynomial time.*
- *Given an equilibrium of M', an equilibrium of M can be constructed in polynomial time.*

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